Universal and nonuniversal renormalizations in Fermi liquids

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We discuss an interplay between the Fermi-liquid (FL) theory and diagrammatic perturbative approach to interacting Fermi systems. In the FL theory for Galilean-invariant systems, mass renormalization m^*/m comes exclusively from fermions at the Fermi surface. We show that in a diagrammatic perturbation theory the same result for m^*/m comes from fermions both at and away from the Fermi surface. The equivalence of the FL and perturbative approaches is based on a particular relation between self-energy contributions from high- and low-energy fermions. We argue that care has to be exercised in the renormalization-group approach to a FL in order not to miss the high-energy contribution to m^*/m. As particular examples, we discuss m^*/m and the quasiparticle residue Z for two-dimensional and three-dimensional systems with both SU(2) and SU(N) symmetries, and with a short-range interaction. We derive an expression for the anisotropic part of the Fermi-liquid vertex in the large-N limit of the SU(N) case.

I. INTRODUCTION

Despite its apparent simplicity, the Landau Fermi-liquid (FL) theory is one of the most nontrivial theories of interacting fermions.[1–5] It states that the linewidth of a state near the Fermi surface (FS) is smaller than its energy so that the quasiparticle propagator $G(\omega,\mathbf{p})$ has a well-defined pole at $\omega = p_F(|\mathbf{p}| - p_F)/m^* + \mathcal{O}(|\mathbf{p}| - p_F)^2$, where p_F is the Fermi momentum. It also states that the quasiparticle residue Z and effective mass m^* are expressed in terms of an interaction vertex $\Gamma^\omega_{\alpha\beta,\gamma\delta}(p,q)$, where $p \equiv (\omega,\mathbf{p})$ and $q \equiv (\omega',\mathbf{q})$, with one of the two "four momenta" on the FS, e.g., $p \equiv (0,\mathbf{p}_F)$, where $\mathbf{p}_F \equiv (\mathbf{p}/|\mathbf{p}|)p_F$. [The vertex $\Gamma^\omega_{\alpha\beta,\gamma\delta}$ is obtained from a fully renormalized, antisymmetrized vertex $\Gamma_{\alpha\beta,\gamma\delta}(p,q;p_1,q_1)$ in the limit of zero momentum- and vanishing energy transfer, i.e., for $|\mathbf{p}_1| = |\mathbf{p}|$, $|\mathbf{q}_1| = |\mathbf{q}|$, $\omega_1 \to \omega$, and $\omega_1' \to \omega'$.] Finally, the FL theory states that, for a Galilean-invariant system ($\mathbf{p}^2/2m$ dispersion for a free particle), which is the only case considered in this paper, the effective mass m^* is expressed via $\Gamma^\omega_{\alpha\beta,\gamma\delta}(p,q)$ with *both* four momenta on the FS. On the other hand, renormalization of the quasiparticle residue Z comes from fermions with $p = p_F$ but q, in general, is away from the FS.

Explicitly, for a Galilean-invariant system,[1,6]

$$G_p = \frac{Z}{\omega - p_F(|\mathbf{p}| - p_F)/m^*}, \quad (1.1)$$

where

$$\frac{1}{Z} = 1 - \frac{i}{2}\sum_{\alpha\beta} \int \Gamma^\omega_{\alpha\beta,\alpha\beta}(p_F,q)(G_q^2)_\omega \frac{d^{D+1}q}{(2\pi)^{D+1}}, \quad (1.2a)$$

$$\frac{1}{m^*} = \frac{1}{m} - A_D \sum_{\alpha\beta} \int \Gamma^\omega_{\alpha\beta,\alpha\beta}(p_F,q_F) \frac{\mathbf{p}_F \cdot \mathbf{q}_F}{p_F^2} d\Omega_q, \quad (1.2b)$$

Ω_q is the solid angle, $A_D = Z^2 k_F^{D-2}/2(2\pi)^D$, G_q is the full fermionic propagator, and $(G_q^2)_\omega$ is the product of the two Green's functions with the same momenta and infinitesimally close frequencies. Note that integration in Eq. (1.2b) is only

over $d\Omega_q$, which implies that mass renormalization comes solely from fermions on the FS. In the field-theoretical language, mass renormalization is then a low-energy, universal phenomenon while a reduction in Z from its bare value of one is a high-energy, nonuniversal phenomenon.

The effective mass m^* and Z factor can also be obtained by expanding the self-energy $\Sigma(\omega,\epsilon_\mathbf{p})$ to first order in ω and $\epsilon_\mathbf{p}$,

$$\Sigma(\omega,\epsilon_\mathbf{p}) = (\omega - \epsilon_\mathbf{p})\left(\frac{1}{Z} - 1\right) - \epsilon_\mathbf{p}\left(\frac{m}{m^*} - 1\right) + \mathcal{O}(\omega^2,\epsilon_p^2). \quad (1.3)$$

[We define Σ by $G_p^{-1} = \omega - \epsilon_\mathbf{p} + \Sigma(\omega,\epsilon_\mathbf{p})$ with $\epsilon_\mathbf{p} = (\mathbf{p}^2 - p_F^2)/2m$.] As, in practice, the self-energy is obtained via a diagrammatic perturbation theory, we will refer to this approach as to "perturbative." In the earlier days of the FL theory, perturbative calculations were used as a check of the general FL relations, e.g., it has been verified[7,8] that the *values* of m^*/m and Z in Eq. (1.3) are the same as in Eqs. (1.2a) and (1.2b). However, whether mass renormalization in Eq. (1.3) comes from low energies, as it does in Eq. (1.2b), has not been verified.

In this paper we demonstrate that, in a diagrammatic calculation, mass renormalization *is not*, in general, a low-energy phenomenon. A low-energy contribution to m^* does, indeed, exist but there is also another, high-energy contribution. Only the sum of the two contributions reproduces the Landau formula for the effective mass, Eq. (1.2a). There are situations (see below) when the high-energy contribution is relatively small but, in general, it is of the same order as the low-energy one.

The reason why low-energy mass renormalization is generally not the full result in a diagrammatic calculation, can be traced back to the fact that the building block of diagrammatics is a nonantisymmetrized interaction potential $U(|\mathbf{k}|)$ rather than the antisymmetrized vertex function Γ^ω. An expression for Γ^ω in terms of $U(|\mathbf{k}|)$ does contain a high-energy contribution and, when the self-energy is expressed in terms

of $U(|\mathbf{k}|)$ rather than in terms of Γ^ω, these high-energy terms do contribute to the effective mass. When one re-expresses Σ in terms of Γ^ω, the high-energy contributions to m^* cancel out in a Galilean-invariant system. When Galilean invariance is broken, mass renormalization has a contribution from high energies already in a FL theory.

An issue where mass renormalization comes from is important for the interpretation of a Galilean-invariant Fermi liquid as a fixed point of the momentum-space renormalization-group (RG) transformation.[9–11] In the RG approach, one progressively integrates out high-energy fermions ending up with a renormalized interaction among low-energy ones. According to Eq. (1.2b), this interaction is all one needs to evaluate the effective mass. Our finding that, in a diagrammatic calculation, m^*/m has contributions from both low- and high-energy fermions implies that care has to be exercised in applying the RG approach to a FL. Specifically, we argue that to recover the Landau formula for m^*/m, one also has to take into account that, in the process of RG flow, the "bare" mass for the low-energy theory changes from the free-fermion mass m to a different value (m_B). The difference $m_B/m-1$ comes from high energies. Only the sum of regular renormalization from m to m_B and low-energy renormalization of m_B yields the agreement with the FL theory.

We further demonstrate that there exists a certain identity [cf. Eq. (4.1)], which relates the high- and low-energy terms. This identity involves combinations of fermionic Green's functions in particle-hole and particle-particle channels, and is exact to first order in ω and $\epsilon_\mathbf{p}$. Adding this identity to the diagrammatic self-energy $\Sigma(\omega, \epsilon_\mathbf{p})$ does not change $\mathcal{O}(\omega)$ and $\mathcal{O}(\epsilon_\mathbf{p})$ terms in Σ, i.e., it does not change Z and m^*/m but, at the same time, it transforms the high-energy contribution into the low-energy one and makes the diagrammatic self-energy equivalent to the self-energy extracted from the FL theory.

An interesting example of comparison between the FL and perturbative approaches is the large-N limit for an SU(N)-invariant two-dimensional (2D) system with short-range interaction. The perturbative self-energy in this case is obtained simply by retaining the random-phase approximation (RPA) diagrams with maximal number of particle-hole bubbles at each order in the interaction. It is not enough, however, to retain only diagrams with a maximal number of bubbles in order to construct Γ^ω because these diagrams contribute only to an isotropic part of Γ^ω and, therefore, do not lead to mass renormalization. We show that the perturbation theory for an anisotropic part of Γ^ω can be resummed to infinite order in U even for subleading in $1/N$ terms and the resulting expression for m^* coincides with that obtained from the self-energy.

The structure of this paper is as follows. In Sec. II, we discuss the FL theory and perturbation series for Γ^ω. We briefly discuss the three-dimensional (3D) case and present the FL expressions for m^*/m and Z in 2D with a short-range interaction (to the best of our knowledge, the result for Z has not been derived in the prior literature). In Sec. III, we obtain the self-energy in the diagrammatic perturbation theory both in 3D and 2D and identify the low- and high-energy contributions to the effective mass. We show that m^*/m ad Z are

indeed the same as in the FL theory but at least part of mass renormalization comes from high energies. Moreover, we show that, in 2D, entire mass renormalization to second order in the interaction comes from high energies, if the calculation is performed by combining internal fermions into particle-hole pairs, while the high-energy part is twice larger and of opposite sign to the low-energy part, if internal fermions are combined into particle-particle pairs. In Sec. IV, we reconcile the two approaches by proving a particular relation between the convolutions of Green's functions. In Sec. V we discuss an extension of our results to the SU(N) case and consider the large N limit. Finally, in Sec. VI we present our conclusions.

II. FERMI-LIQUID THEORY

A. Pitaevskii-Landau relations

We remind the reader that Eqs. (1.2a) and (1.2b) in the FL theory are based on the Pitaevskii-Landau relations—the three identities for the derivatives of the Green's function,[1,6]

$$\frac{\partial G_p^{-1}}{\partial \omega} = \frac{1}{Z} = 1 - \frac{i}{2}\sum_{\alpha\beta}\int \Gamma^\omega_{\alpha\beta,\alpha\beta}(p_F,q)(G_q^2)_\omega \frac{d^{D+1}q}{(2\pi)^{D+1}},$$

(2.1)

$$\begin{aligned}\mathbf{p}_F\frac{\partial G_p^{-1}}{\partial \mathbf{p}} &= -\frac{p_F^2}{m^*Z}\\ &= -\frac{p_F^2}{m} + \frac{i}{2}\sum_{\alpha\beta}\int \Gamma^k_{\alpha\beta,\alpha\beta}(p_F,q)\frac{\mathbf{p}_F\cdot\mathbf{q}}{m}(G_q^2)_k \frac{d^{D+1}q}{(2\pi)^{D+1}},\end{aligned}$$

(2.2)

$$\frac{1}{Z} = 1 - \frac{i}{2}\sum_{\alpha\beta}\int \Gamma^\omega_{\alpha\beta,\alpha\beta}(p_F,q)(G_q^2)_\omega \frac{\mathbf{p}_F\cdot\mathbf{q}}{p_F^2} \frac{d^{D+1}q}{(2\pi)^{D+1}}.$$

(2.3)

The first two relations originate from particle-number conservation while the third relation is a consequence of Galilean invariance. In Eq. (2.2), the object $(G_q^2)_k$ is the product of two Green's functions with the same frequencies and infinitesimally close momenta, and $\Gamma^k_{\alpha\beta,\gamma\delta}$ is the vertex in the limit of zero-frequency transfer and vanishing momentum transfer. The latter is related to $\Gamma^\omega_{\alpha\beta,\gamma\delta}$ by an integral equation

$$\begin{aligned}\Gamma^k_{\alpha\beta,\alpha\beta}(p,q) &= \Gamma^\omega_{\alpha\beta,\alpha\beta}(p,q)\\ &- \frac{k_F^{D-1}Z^2}{v_F(2\pi)^D}\sum_{\xi,\eta}\int \Gamma^\omega_{\alpha\xi,\alpha\eta}(p,q')\Gamma^k_{\eta\beta,\xi\beta}(q',q)d\Omega_{q'}.\end{aligned}$$

(2.4)

In addition, $(G_q^2)_k$ is related to $(G_q^2)_\omega$ by

$$(G_q^2)_k - (G_q^2)_\omega \equiv \delta G_q^2 = -\frac{2\pi i Z^2 m^*}{p_F}\delta(\omega)\delta(|\mathbf{q}|-p_F).$$

(2.5)

Note that Eqs. (2.1) and (2.2) contain the integrals over all intermediate states with momenta q. However, using the ad-
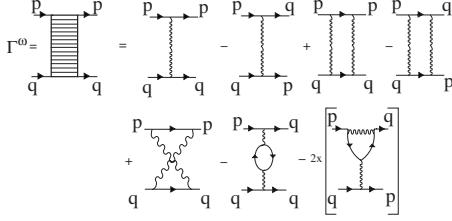
FIG. 1. First- and second-order diagrams for the Fermi-liquid vertex $\Gamma^\omega_{\alpha\beta,\gamma\delta}(p,q)$. The initial four momenta p and q are associated with spin projections α and β, respectively. The final four momenta p and q are associated with spin projections γ and δ, respectively.

ditional property of Galilean invariance [Eq. (2.3)], one can eliminate the high-energy contribution to m^* (but not to Z). Indeed, substituting Eqs. (2.4) and (2.5) into Eq. (2.2) and using Eq. (2.3), one reduces Eq. (2.2) to

$$
\begin{aligned}
\mathbf{p}_F \frac{\partial G_p^{-1}}{\partial \mathbf{p}} &= -\frac{p_F^2}{m^* Z} = -\frac{p_F^2}{mZ} + \frac{i}{2} \frac{p_F^2}{m^* Z} \\
&\quad \times \sum_{\alpha\beta} \int \Gamma^\omega_{\alpha\beta,\alpha\beta}(p_F,q) \delta G_q^2 \frac{\mathbf{p}_F \cdot \mathbf{q}}{p_F^2} \frac{d^{D+1}q}{(2\pi)^{D+1}} \\
&= -\frac{p_F^2}{mZ} + \frac{p_F^2}{Z} A_D \sum_{\alpha\beta} \int \Gamma^\omega_{\alpha\beta,\alpha\beta}(p_F,q_F) \frac{\mathbf{p}_F \cdot \mathbf{q}_F}{p_F^2} d\Omega_q,
\end{aligned}
\tag{2.6}
$$

which is equivalent to Eq. (1.2b) for mass renormalization. We emphasize again that Eq. (1.2b), which involves only low-energy fermions, is based not only on particle-number conservation [Eqs. (2.4) and (2.5)] but also on Eq. (2.3), specific to Gallilean-invariant systems.

Combining Eqs. (2.1), (2.3), and (2.6), one can construct the self-energy to first order in ω and $\epsilon_{\mathbf{p}}$ as

$$
\begin{aligned}
\Sigma_{\mathrm{FL}}(\omega, \epsilon_{\mathbf{p}}) &= (\omega - \epsilon_{\mathbf{p}}) \left[-\frac{i}{2} \sum_{\alpha\beta} \int \Gamma^\omega_{\alpha\beta,\alpha\beta}(p_F,q) G_q^2 \frac{d^{D+1}q}{(2\pi)^{D+1}} \right] \\
&\quad + \epsilon_{\mathbf{p}} \left[\frac{i}{2Z} \sum_{\alpha\beta} \int \Gamma^\omega_{\alpha\beta,\alpha\beta}(p_F,q) \frac{\mathbf{p}_F \cdot \mathbf{q}}{p_F^2} \delta G_q^2 \frac{d^{D+1}q}{(2\pi)^{D+1}} \right].
\end{aligned}
\tag{2.7}
$$

B. Perturbation theory for Γ^ω

The vertex Γ^ω can be obtained via a perturbative expansion in $U(|\mathbf{k}|)$. Diagrams for Γ^ω to second order in $U(|\mathbf{k}|)$ are presented in Fig. 1. We emphasize that (i) Γ^ω is antisymmetrized vertex and (ii) that the renormalizations in particle-hole and particle-particle channels are both relevant. Restricting with only the renormalization in the particle-hole channel, as it was done in, e.g., Ref. 11, leads to incorrect results.

Assume first that $U(|\mathbf{k}|)=\mathrm{const}\equiv U$ (contact interaction). In this case,

$$
\begin{aligned}
&\Gamma^\omega_{\alpha\beta,\gamma\delta}(p_F,q) \\
&= \delta_{\alpha\gamma}\delta_{\beta\delta} \left[U + iU^2 \int (G_l G_{q-p_F+l} + G_l G_{q+p_F-l}) \frac{d^{D+1}l}{(2\pi)^{D+1}} \right] \\
&\quad - \delta_{\alpha\delta}\delta_{\beta\gamma} \left[U + iU^2 \int G_l G_{q+p_F-l} \frac{d^{D+1}l}{(2\pi)^{D+1}} \right].
\end{aligned}
\tag{2.8}
$$

The first term in Eq. (2.8) is the renormalized interaction with zero-momentum transfer and the second term is obtained by antisymmetrization. We see that the first ("direct") term contains contributions from both the particle-hole and particle-particle channels while the second ("exchange") term contains only a contribution from the particle-particle channel.

In 3D, explicit expression for Γ^ω when both particles are on the FS (i.e., $q=q_F$) were obtained long time ago (see Refs. 1 and 7). A similar calculation, for the 2D case, yields[12]

$$
\begin{aligned}
\Gamma^\omega_{\alpha\beta,\gamma\delta}(p_F,q_F) &= \Gamma^\omega_{\alpha\beta,\gamma\delta}(\theta) \\
&= \frac{1}{2} \delta_{\alpha\gamma}\delta_{\beta\delta} \left[U + \frac{U^2 m}{2\pi} \ln\left(\cos\frac{\theta}{2}\right) \cdots \right] \\
&\quad - \frac{1}{2} \boldsymbol{\sigma}_{\alpha\gamma} \cdot \boldsymbol{\sigma}_{\beta\delta} \left[U + \frac{U^2 m}{2\pi} \ln\left(\cos\frac{\theta}{2}\right) + \cdots \right],
\end{aligned}
\tag{2.9}
$$

where θ is the angle between $\mathbf{p}_F$ and $\mathbf{q}_F$ and dots stand for angle-independent U^2 terms and terms of higher orders in U. In deriving Eq. (2.9), we used a result for the static particle-particle bubble in 2D

$$
\Pi_{pp}(\omega = 0, \ |\mathbf{k}| \leq 2p_F) = i \int \frac{d^3 l}{(2\pi)^3} G_l G_{k-l}
$$

$$
= \frac{m}{2\pi} \ln \frac{2p_F}{|\mathbf{k}|}
\tag{2.10}
$$

(up to an irrelevant constant). Note that in 2D, the angular dependence of $\Gamma^\omega(\theta)$, which is responsible for mass renormalization, comes entirely from the interaction in the particle-particle channel, i.e., from $\Pi_{pp}(\theta=0,|\mathbf{k}|<2p_F)$. Since the 2D particle-hole bubble $\Pi_{ph}(k)=i\int d^3 l G_l G_{l+k}/(2\pi)^3$ is independent of $|\mathbf{k}|$ for $|\mathbf{k}| \leq 2p_F$, renormalization of the interaction in the particle-hole channel only adds a constant to U and is, therefore, irrelevant for m^*. The formula for $\Gamma^\omega_{\alpha\beta,\gamma\delta}(p_F,q)$ for q is away from the FS is rather complex and we refrain from presenting it.

C. Effective mass and quasiparticle residue

Substituting Γ^ω from Eq. (2.8) into Eqs. (1.2a) and (1.2b) and evaluating the integrals in 3D, we reproduce the known results for m^*/m (Refs. 7 and 8),

$$
\frac{m^*}{m} = 1 + \left(\frac{8}{15}\right)(7 \ln 2 - 1) \left(\frac{mUp_F}{4\pi^2}\right)^2
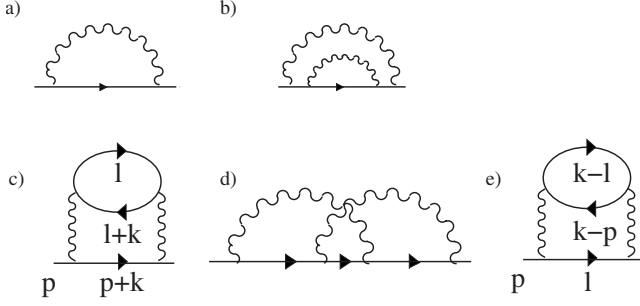\tag{2.11}
$$

and Z (Refs. 8 and 13),

FIG. 2. First- and second-order diagrams for the fermionic self-energy $\Sigma_{\text{pert}}(\omega, \epsilon_{\mathbf{p}})$. For a momentum-independent interaction $U(|\mathbf{q}|) = U$, only second- and higher order diagrams renormalize the mass and Z. For a momentum-dependent interaction, mass renormalization starts already at the first order. Diagram (e) is the same as (c), except for internal fermions are combined into a particle-particle rather than a particle-hole pair.

$$Z = 1 - 8 \ln 2 \left(\frac{m U p_F}{4 \pi^2} \right)^2. \qquad (2.12)$$

In 2D, Eq. (2.9) immediately gives[12]

$$\frac{m^*}{m} = 1 + \frac{1}{2} \left(\frac{m U}{2 \pi} \right)^2 \qquad (2.13)$$

while for Z we obtain after numerical integration of $\Gamma^{\omega}_{\alpha\beta,\gamma\delta}(p_F, q)$ instead of Eq. (2.9)

$$Z \approx 1 - C \left(\frac{m U}{2 \pi} \right)^2, \qquad (2.14)$$

where $C = 0.6931 \cdots$. To high numerical accuracy, C is equal to $\ln 2$ but we did not attempt to prove this analytically. We remind that, in the FL formulation, m^*/m comes exclusively from the interaction between particles on the FS.

Momentum-dependent interaction

For a momentum-dependent interaction $U(|\mathbf{k}|)$, expressions for m^*/m and Z are generally more complex. Mass renormalization now occurs already at the first order in $U(|q|)$. In 3D,

$$\frac{m^*}{m} = 1 - \frac{m p_F}{16 \pi^2} \int_0^2 dz z \{ U[p_F(2 - z)^{1/2}] - U[p_F(2 + z)^{1/2}] \}$$
$$+ O(U^2). \qquad (2.15)$$

Renormalization of Z still occurs beginning from the second order in $U(|\mathbf{k}|)$.

III. PERTURBATION THEORY FOR THE SELF-ENERGY

We now discuss how m^*/m and Z occur in the diagrammatic perturbation theory. Again, we consider first the case of a constant interaction U; a momentum-dependent interaction will be discussed later. The first-order term in the self-energy [Fig. 2(a)] is irrelevant in this case and we focus on the second-order diagrams in Fig. 2. Diagram (b) just shifts the chemical potential and is also irrelevant, so we need to

consider only diagrams (c) and (d). Relabeling the fermionic momenta, it is easy to see that, for a constant U, diagram (d) is equal to $-1/2$ of diagram (c), so there is essentially one second-order diagram to be considered, e.g., diagram (c). This diagram contains three Green's functions, two of which share a common internal momentum. Labeling the momenta as shown in diagram (c) and integrating over the internal four-momentum l, we end up with a particle-hole bubble. Alternatively, labeling the momenta as shown in diagram (e) and integrating over l, we end up with a particle-particle bubble.

We start with combining two internal fermions into a particle-hole bubble; the particle-particle combination is discussed in Sec. III B 2. Subtracting from the particle-hole form of $\Sigma(\omega, \epsilon_{\mathbf{p}})$ its value at $\omega = 0$, $\epsilon_{\mathbf{p}} = 0$, we find

$$\Sigma_{\text{pert}}(\omega, \epsilon_{\mathbf{p}}) - \Sigma(0, 0) = -U^2 \int G_l G_{k - p_F + l}(G_{k + \epsilon} - G_k) d_{lk}, \qquad (3.1)$$

where $d_{lk} \equiv d^{D+1} l \, d^{D+1} k / (2 \pi)^{2(D+1)}$ and

$$\epsilon = \left(\omega, \frac{\epsilon_{\mathbf{p}}}{v_F p_F} \mathbf{p}_F \right) \qquad (3.2)$$

is the (small) external four-momentum. The self-energy can be further split into two parts as

$$\Sigma_{\text{pert}}(\omega, \epsilon_{\mathbf{p}}) - \Sigma(0, 0) = \delta\Sigma_1(\omega, \epsilon_{\mathbf{p}}) + \delta\Sigma_2(\omega, \epsilon_{\mathbf{p}}), \qquad (3.3)$$

where

$$\delta\Sigma_1(\omega, \epsilon_{\mathbf{p}}) = U^2 \int G_l G_{k - p_F + l} G_q^2 \left(\omega - \epsilon_{\mathbf{p}} \frac{\mathbf{p}_F \cdot \mathbf{k}}{p_F^2} \right) d_{lk}, \qquad (3.4a)$$

$$\delta\Sigma_2(\omega, \epsilon_{\mathbf{p}}) = -U^2 \int' G_l G_{k - p_F + l}(G_{k - \epsilon} - G_k) d_{lk}. \qquad (3.4b)$$

The difference between the two parts is as follows. In the first part, the integrand was expanded to first order in ω and $\epsilon_{\mathbf{p}}$. This is justified if typical internal energies remain finite when $\omega, \epsilon_{\mathbf{p}} \to 0$. This is a regular, high-energy contribution to the self-energy coming from fermions not confined to the FS. The second term is an anomalous contribution from internal energies of order ω and $\epsilon_{\mathbf{p}}$, which cannot be obtained by an expansion of Σ_{pert} in the external energies. This second term is a low-energy contribution (to emphasize this, we put a prime on the integral for this part).

A. 3D case

Evaluating the integrals in Eqs. (3.4a) and (3.4b) for the 3D case, we find that both contributions are finite, namely,

$$\delta\Sigma_1(\omega, \epsilon_{\mathbf{p}}) = 8 \ln 2 \left(\frac{m U p_F}{4 \pi^2} \right)^2 (\omega - \epsilon_{\mathbf{p}}) + \frac{4}{3} \epsilon_{\mathbf{p}} (4 \ln 2 - 1)$$
$$\times \left(\frac{m U p_F}{4 \pi^2} \right)^2,$$

$$\delta\Sigma_2(\omega,\epsilon_\mathbf{p}) = -\frac{4}{5}\epsilon_\mathbf{p}(2\ln 2-1)\left(\frac{mUp_F}{4\pi^2}\right)^2. \quad (3.5)$$

Adding up two parts and casting the result into the form of Eq. (1.3), we recover the Galitskii's result (Ref. 8),

$$\Sigma_{\text{pert}}(\omega,\epsilon_\mathbf{p}) - \Sigma(0,0) = \left(\frac{mUp_F}{4\pi^2}\right)^2\left[8\ln 2(\omega-\epsilon_\mathbf{p}) + \frac{8}{15}(7\ln 2-1)\epsilon_\mathbf{p}\right]. \quad (3.6)$$

This self-energy indeed produces the same m^*/m and Z as in the FL theory, Eqs. (2.12) and (2.11), We see, however, that mass renormalization—determined by a stand-alone $\epsilon_\mathbf{p}$ term in the self-energy—comes from both the high- and low-energy parts of Σ_{pert}. Only the sum of the two contributions recovers the FL formula for m^*/m. On the other hand, renormalization of Z comes only from $\delta\Sigma_1$, i.e., from high energies.

B. 2D case

The difference between the FL and diagrammatic approaches becomes even more obvious in 2D. Since, to the best of our knowledge, the Z factor for a 2D Fermi liquid with a short-range interaction has not been calculated before, we consider the 2D case in more detail. We also use the 2D case as an example to show that an interplay between high-energy and low-energy contributions to the effective mass (but not the full result) depends on whether the self-energy is calculated via particle-hole or particle-particle bubbles.

1. Perturbative self-energy via particle-hole bubble

The calculation of the perturbative self-energy via a particle-hole bubble is based on Eqs. (3.4a) and (3.4b). It 2D, the particle-hole bubble can be found analytically for any ω and $|\mathbf{k}|$,

$$\Pi_{\text{ph}}(\omega,\mathbf{k}) = -\frac{m}{2\pi}$$
$$\times\left[1 + i\frac{\sqrt{2}\tilde{\omega}}{\sqrt{\tilde{k}^2-\tilde{k}^4-\tilde{\omega}^2+\sqrt{(\tilde{k}^2-\tilde{k}^4-\tilde{\omega}^2)^2-4\tilde{\omega}^2\tilde{k}^4}}}\right], \quad (3.7)$$

where $\tilde{\omega}=2\omega m/p_F^2$ and $\tilde{k}=|\mathbf{k}|/2p_F$. To calculate the regular part of the self-energy, one needs to know the entire bubble while the anomalous part is determined only by the static bubble $\Pi_{\text{ph}}(0,|\mathbf{k}|)$. Performing the angular integral in $\delta\Sigma_1$ analytically and remaining integrals numerically, and all integrals in $\delta\Sigma_2$ analytically, we obtain

$$\delta\Sigma_1(\omega,\epsilon_\mathbf{p}) = C\left(\frac{mU}{2\pi}\right)^2(\omega-\epsilon_\mathbf{p}) + \frac{\epsilon_\mathbf{p}}{2}\left(\frac{mU}{2\pi}\right)^2, \quad (3.8a)$$

$$\delta\Sigma_2(\omega,\epsilon_\mathbf{p}) = 0 \quad (3.8b)$$

with

$$C = \frac{1}{2\pi}\int_0^\infty dx\int_0^\infty dy\frac{\partial}{\partial y}\frac{\sqrt{A+\sqrt{A^2+x^2y^2}}}{\sqrt{A^2+x^2y^2}}F(x,y), \quad (3.9)$$

$A=(x-x^2+y^2)/2$, and

$$F(x,y) = \frac{y}{\sqrt{A+\sqrt{A^2+x^2y^2}}} - \theta(x-1)\sqrt{\frac{x-1}{x}}. \quad (3.10)$$

Numerical integration yields, to high accuracy, $C=0.6931\cdots=\ln 2$; same as for the prefactor in Eq. (2.14).

The reason why $\delta\Sigma_2=0$ in 2D is very simple. This contribution is expressed via a static particle-hole bubble as

$$\delta\Sigma_2(\omega,\epsilon_\mathbf{p}) = i\frac{\epsilon_\mathbf{p}U^2}{2\pi^2 v_F}\int_0^{2p_F}d|\mathbf{k}|\Pi_{\text{ph}}(\omega=0,|\mathbf{k}|)$$
$$\times\frac{1-|\mathbf{k}|^2/2p_F^2}{\sqrt{1-(|\mathbf{k}|/2p_F)^2}}. \quad (3.11)$$

As $\Pi_{\text{ph}}(\omega=0,|\mathbf{k}|)$ is independent of $|\mathbf{k}|$ for $|\mathbf{k}|\leq 2p_F$, the integral over $|\mathbf{k}|$ vanishes. For the same reason, mass renormalization in the FL theory comes only from the particle-particle part of the vertex in Eq. (2.9).

Casting the result into the form of Eq. (1.3), we again reproduce the FL results for m^*/m and Z, Eqs. (2.13) and (2.14). However, we see that now m^*/m comes solely from the high-energy part of the self-energy, in an (apparent) contradiction to the FL theory, where it comes from low energies.

2. Perturbative self-energy via particle-particle bubble

We now show that an interplay between high- and low-energy contributions to the perturbative self-energy depends on the way how we obtain it. To demonstrate this, we obtain the same perturbative self-energy $\Sigma_{\text{pert}}(\omega,\epsilon_p)$ as in Eq. (3.8a) by combining internal fermions into particle-particle rather than particle-hole pairs. We will see that in this situation mass renormalization comes from both high and low energies.

For simplicity, we consider a momentum-independent interaction in 2D and restrict attention to the second order in U. Labeling the momenta as shown in diagram (e) of Fig. 2, we obtain for the second-order self-energy

$$\tilde{\Sigma}_{\text{pert}}(\omega,\epsilon_\mathbf{p}) - \tilde{\Sigma}(0,0) = -U^2\int G_l G_{k+p_F-l}(G_{k-\epsilon}-G_k)d_{lk}. \quad (3.12)$$

We denote the self-energy obtained in this way as $\tilde{\Sigma}_{\text{pert}}$ to distinguish it from the self-energy in the particle-hole form. As before, we split the difference $\tilde{\Sigma}_{\text{pert}}(\omega,\epsilon_\mathbf{p})-\tilde{\Sigma}(0,0)$ into a sum of regular and anomalous contributions as

$$\delta\tilde{\Sigma}_1(\omega,\epsilon_\mathbf{p}) = U^2\int G_l G_{k+p_F-l}G_k^2\left(\omega-\epsilon_\mathbf{p}\frac{\mathbf{p}_F\cdot\mathbf{k}}{p_F^2}\right)d_{lk}, \quad (3.13a)$$

$$\delta\widetilde{\Sigma}_2(\omega,\epsilon_{\mathbf{p}}) = -U^2 \int' G_l G_{k+p_F-l}(G_{k-\epsilon} - G_k) d_{lk}.$$
(3.13b)

The anomalous part $\delta\widetilde{\Sigma}_2$ is now expressed via a static *particle-particle* bubble as

$$\delta\widetilde{\Sigma}_2(\omega,\epsilon_{\mathbf{p}}) = -\frac{\epsilon_{\mathbf{p}} U^2}{2\pi^2 v_F} \int_0^{2p_F} d|\mathbf{k}| \Pi_{\mathrm{pp}}(\omega=0,|\mathbf{k}|)$$
$$\times \frac{1 - |\mathbf{k}|^2/2p_F^2}{\sqrt{1 - (|\mathbf{k}|/2p_F)^2}},$$
(3.14)

where $\Pi_{\mathrm{pp}}(\omega=0,|\mathbf{k}|)$ is given in Eq. (2.10). In contrast to Eq. (3.11), the integral over $|\mathbf{k}|$ now does not vanish and we obtain

$$\delta\widetilde{\Sigma}_2(\omega,\epsilon_{\mathbf{p}}) = -\frac{\epsilon_{\mathbf{p}}}{2}\left(\frac{mU}{2\pi}\right)^2.$$
(3.15)

Therefore, in contrast to the particle-hole case, the low-energy contribution to mass renormalization in the particle-particle case is finite but opposite in sign to mass renormalization in the FL theory, Eq. (2.13).

The regular part of self-energy, $\delta\widetilde{\Sigma}_1$, renormalizes both the Z factor and effective mass. To evaluate this part of the self-energy, a static approximation for Π_{pp} is not sufficient and we need a dynamic form of the particle-particle propagator. Re-expressing $\delta\widetilde{\Sigma}_1$ in Eq. (3.13a) via the particle-particle polarization bubble and shifting the momentum as $k - p_F \to k$, we obtain

$$\delta\widetilde{\Sigma}_1(\omega,\epsilon_{\mathbf{p}}) = U^2 \int \frac{\Pi_{\mathrm{pp}}(\omega_k,\mathbf{k})}{(\omega_k - \epsilon_{k-p_F})^2}$$
$$\times \left[(\omega - \epsilon_{\mathbf{p}}) + \epsilon_{\mathbf{p}}\left(2 - \frac{\mathbf{p}_F \cdot \mathbf{k}}{p_F^2}\right)\right] d_k.$$
(3.16)

We remind that $d_k = k\,dk\,d\theta_k\,d\omega_k/(2\pi)^3$. If we approximated $\Pi_{\mathrm{pp}}(\omega_k,\mathbf{k})$ by its static form, $\delta\widetilde{\Sigma}_1$ would vanish after integration over ω_k because of the double pole. The dynamic $\Pi_{\mathrm{pp}}(\omega_k,\mathbf{k})$, however, has branch cuts in both upper and lower half planes of ω_k, which ensures that the frequency integral is nonzero.

The dynamic polarization bubble is obtained by standard means and, for $|\mathbf{k}| < 2p_F$, is given by

$$\Pi_{\mathrm{pp}}(\omega_k,\mathbf{k}) = \frac{m}{2\pi^2} \int_0^{\pi/2} d\phi \ln S_\phi(\omega_k,\mathbf{k}),$$
(3.17)

where

$$S_\phi(\omega_k,\mathbf{k}) = \frac{m\omega_k - p_F^2 + \mathbf{k}^2/4}{m\omega_k - |\mathbf{k}|\cos\phi\sqrt{p_F^2 + \frac{\mathbf{k}^2}{4}\sin^2\phi} - \frac{\mathbf{k}^2}{2}\cos^2\phi}$$
$$\times \frac{p_F^2}{m\omega_k + |\mathbf{k}|\cos\phi\sqrt{p_F^2 - \frac{\mathbf{k}^2}{4}\sin^2\phi} + \frac{\mathbf{k}^2}{2}\cos^2\phi}.$$
(3.18)

For $|\mathbf{k}| > 2p_F$ the expression for Π_{pp} is more complex but we do not need it as in this region the pole and branch cut are in the same half plane of ω_k and the frequency integral vanishes.

Substituting Eq. (3.17) into Eq. (3.16), evaluating the frequency integral over a half plane where there is no double pole (this requires separate considerations for $|\mathbf{k}| > 2p_F \cos\theta_k$ and $|\mathbf{k}| < 2p_F \cos\theta_k$), and evaluating the remaining integrals over ϕ, θ_k, and $|\mathbf{k}|$ numerically, we obtain

$$\delta\widetilde{\Sigma}_1(\omega,\epsilon_{\mathbf{p}}) = C\left(\frac{mU}{2\pi}\right)^2(\omega - \epsilon_{\mathbf{p}}) + \epsilon_{\mathbf{p}}\left(\frac{mU}{2\pi}\right)^2.$$
(3.19)

where, as before, $C = 0.6931\cdots = \ln 2$.

The total particle-particle self-energy, given by the sum of Eqs. (3.15) and (3.19), is indeed the same as the total particle-hole self-energy, given by the sum of Eqs. (3.8a) and (3.8b), and total mass renormalization is the same as in Eq. (2.13). However, we see that mass renormalization now comes from both high and low energies.

C. Momentum-dependent interaction

For a momentum-dependent interaction $U(|\mathbf{k}|)$, the difference between the diagrammatic and FL formulas for the self-energy is less drastic. In particular, for a weak but momentum-dependent interaction, mass renormalization to first order in $U(|\mathbf{k}|)$, as defined by Eq. (2.15), comes only from fermions on the FS.

For completeness, we also present perturbative results for m^*/m and Z for the Coulomb interaction. At small r_s, both m^*/m and Z are quasilinear in r_s for small r_s. In 3D, $m^*/m = 1 - (r_s/2\pi)(4/9\pi)^{1/3}\ln r_s^{-1}$ and $Z = 1 - 0.17696 r_s$ (Ref. 14). In 2D, $m^*/m = 1 - (r_s/\pi\sqrt{2})\ln r_s^{-1}$ (Ref. 14) and $Z = 1 - (r_s/\sqrt{2})(1/2 + 1/\pi)$ (Ref. 15). The case of Coulomb interaction is special in that Z comes from fermions in the vicinity of the FS.

IV. RECONCILIATION OF THE FERMI-LIQUID AND PERTURBATIVE APPROACHES

We have shown in the previous sections that while mass renormalization comes from low energies in the FL theory, it generally contains both high- and low-energy contributions in a diagrammatic perturbation theory. In Secs. IV A–IV C, we show how to reconcile the two approaches.

A. Equivalence of the Fermi-liquid and perturbative approaches for a momentum-independent interaction

To begin, we emphasize that the results for the self-energy in the FL and perturbative approaches need not to coincide

identically because Σ_{FL} given by Eq. (2.7) is only an expansion of the full self-energy to first order in ω and $\epsilon_{\mathbf{p}}$ while Σ_{pert} given by Eqs. (3.3), (3.4a), and (3.4b) contains all orders in ω and $\epsilon_{\mathbf{p}}$. However, Σ_{pert} to first order in ω and $\epsilon_{\mathbf{p}}$ must coincide with Σ_{FL}.

Comparing the two self-energies, we see the difference: while Σ_{FL}, expressed via Γ^{ω}, contains both particle-hole and particles-particle bubbles, Σ_{pert} contains only a particle-hole bubble. We now show that there exists a particular relation between the combinations of the Green's functions which involve particle-hole and particle-particle bubbles, namely,

$$\int (G_l G_{k-p_F+l} + G_l G_{k+p_F-l})(G_{k-\epsilon} - G_k)d_{kl} = 0, \quad (4.1)$$

where ϵ is given by Eq. (3.2). This relation is valid to first order in ϵ (i.e., in ω and $\epsilon_{\mathbf{p}}$) and follows from the identity, which we have already used implicitly when diagram (c) in Fig. 2 was replaced by diagram (e),

$$\int d_{kl} G_l G_{k-p_F+l}(G_{k+\epsilon} - G_k) = \int d_{kl} G_l G_{k+p_F-l}(G_{k-\epsilon} - G_k). \quad (4.2)$$

The identity in Eq. (4.2) is proven by relabeling the four momenta, e.g., by relabeling the momenta in both terms in the second line as $k \to k-p_F+l+\epsilon$ and then relabeling $k+\epsilon \to k$ in the last term. Equation (4.1) follows from Eq. (4.2) once $G_{k+\epsilon} - G_k$ in the first line of Eq. (4.2) is replaced by $G_k - G_{k-\epsilon} + \mathcal{O}(\epsilon^2)$. Adding Eq. (4.1) to $\Sigma_{\mathrm{pert}}(\omega, \epsilon_{\mathbf{p}})$, we find, after simple algebra, that it becomes equal to Σ_{FL}, i.e., the expressions for m^*/m and Z become exactly the same as in Fermi-liquid theory. Analyzing further the left-hand side of Eq. (4.1), we find that both the particle-particle and particle-hole terms contain regular (high-energy) and anomalous (low-energy) contributions. Anomalous contributions contain only the single-particle dispersion $\epsilon_{\mathbf{p}}$ while regular contributions contain both $\epsilon_{\mathbf{p}}$ and ω terms. Expanding the regular contributions to first order in ω and $\epsilon_{\mathbf{p}}$ and equating the prefactors of $\epsilon_{\mathbf{p}}$ and ω terms, we obtain

$$\omega \int d_{kl}(G_l G_{k-p_F+l} + G_l G_{k+p_F-l})G_k^2 = 0, \quad (4.3)$$

$$\epsilon_{\mathbf{p}} \int d_{kl}(G_l G_{k-p_F+l} + G_l G_{k+p_F-l})G_k^2 \frac{\mathbf{p}_F \cdot \mathbf{k}}{p_F^2}$$

$$= -\epsilon_{\mathbf{p}} \int d_{kl}(G_l G_{k-p_F+l} + G_l G_{k+p_F-l})\delta G_k^2 \frac{\mathbf{p}_F \cdot \mathbf{k}}{p_F^2}, \quad (4.4)$$

where δG_k^2, defined by Eq. (2.5), projects the integral over k onto the FS. The left- and right-hand sides of Eq. (4.4) are anomalous (low-energy) and regular (high-energy) contributions, respectively.

We see from Eq. (4.3) that the addition of Eq. (4.1) to Σ_{pert} does not change the result for the ω term in a sense that there is no interplay between high-and low-energy contributions, and Eq. (4.3) simply adds zero to the high-energy contribution. This explains why the ω terms in Σ_{FL} and Σ_{pert} are

identical. On the other hand, by adding Eq. (4.1) to Σ_{pert} we are changing the interplay between the high- and low-energy contributions to the $\epsilon_{\mathbf{p}}$ terms. The regular contribution from Eq. (4.4) cancels the $\epsilon_{\mathbf{p}}$ term in $\delta\Sigma_1$ while the anomalous contribution renders $\delta\Sigma_2$ equal to the FL self-energy, Eq. (2.7).

B. Momentum-dependent interaction

The results of the previous section can be readily extended to the case of a momentum-dependent interaction. In this situation, we obtain, instead of Eqs. (2.8), (3.4a), and (3.4b),

$$\Gamma^{\omega}_{\alpha\beta,\alpha\beta}(p_F, q) = U(0) + i \int \frac{d^{D+1}l}{(2\pi)^{D+1}} U^2(|\mathbf{p}_F - \mathbf{l}|)(G_l G_{q-p_F+l}$$

$$+ G_l G_{q+p_F-l}) - \delta_{\alpha\beta}\left[U(|\mathbf{q} - \mathbf{p}_F|) \right.$$

$$- i \int \frac{d^{D+1}l}{(2\pi)^{D+1}}\{[2U(|\mathbf{q} - \mathbf{p}_F|) - 2U(|\mathbf{q}$$

$$- \mathbf{p}_F|)U(|\mathbf{p}_F - \mathbf{l}|)]G_l G_{l+q-p_F} - U(|\mathbf{p}_F - \mathbf{l}|)$$

$$\left. \times U(|\mathbf{l} - \mathbf{q}|)G_l G_{q+p_F-l}\} \right]$$

and $\delta\Sigma_{\mathrm{pert}}(\omega, \epsilon_{\mathbf{p}}) = \Sigma_{\mathrm{pert}}(\omega, \epsilon_{\mathbf{p}}) - \Sigma_{\mathrm{pert}}(0,0) = \delta\Sigma_1 + \delta\Sigma_2$ with

$$\delta\Sigma_1(\omega, \epsilon_{\mathbf{p}}) = \int d_{lq}[2G_l G_{k-p_F+l}\{U^2(|\mathbf{q} - \mathbf{p}_F|) - 2U(|\mathbf{p}_F$$

$$- \mathbf{l}|)U(|\mathbf{q} - \mathbf{p}_F|)\} - G_l G_{q+p_F-l}U(|\mathbf{p}_F - \mathbf{l}|)$$

$$\times U(|\mathbf{l} - \mathbf{q}|)](G_{k-\epsilon} - G_k),$$

$$\delta\Sigma_2(\omega, \epsilon_{\mathbf{p}}) = -\int d_{lq}[2G_l G_{k-p_F+l}\{U^2(|\mathbf{q} - \mathbf{p}_F|) - 2U(|\mathbf{p}_F$$

$$- \mathbf{l}|)U((|\mathbf{q} - \mathbf{p}_F|)\} - G_l G_{q+p_F-l}U(|\mathbf{p}_F - \mathbf{l}|)$$

$$\times U(|\mathbf{l} - \mathbf{q}|)]G_q^2\left(\omega - \epsilon_{\mathbf{p}}\frac{\mathbf{p}_F \cdot \mathbf{q}}{p_F^2}\right), \quad (4.5)$$

where we neglected first-order terms.

Comparing the expressions for Γ^{ω} and $\delta\Sigma_{\mathrm{pert}}$, we see that Γ^{ω} [and, hence, Σ_{FL} given by Eq. (2.7)] again contains two extra terms not present in the diagrammatic self-energy. These two terms have the same overall factor of $U^2(|\mathbf{p}_F-\mathbf{l}|)$. After some rearranging of the momenta in the products of three fermionic propagators, we obtain an analog of Eq. (4.1) for a momentum-dependent interaction as

$$\int d_{ql} U^2(|\mathbf{p}_F - \mathbf{l}|)(G_l G_{k-p_F+l} + G_l G_{q+p_F-l})(G_{k-\epsilon} - G_k). \quad (4.6)$$

Adding this expression to $\delta\Sigma_{\mathrm{pert}}$, we find after some algebra that high-energy contributions to m^*/m cancel and, to first order in ω and $\epsilon_{\mathbf{p}}$, Σ_{pert} becomes equal to Σ_{FL}.

C. Higher orders of the perturbation theory

So far, we have focused only on the lowest-order perturbation theory. One can show, however, that Eq. (4.6) remains valid if the bare fermionic propagators are replaced by the full ones and $U(|\mathbf{q}|)$ is replaced by a fully renormalized interaction which depends not only on the momentum but also on frequency. This is so because Eq. (4.6) is proven simply by rearranging internal four momenta. Next, the full perturbative self-energy is also obtained from the second-order result, Eq. (4.5), by dressing the interactions and propagators. Adding Eq. (4.6) to the full perturbative self-energy, expressed via the full propagators and full interactions, we immediately recover Σ_{FL} simply because the previous proof of this statement did not rely on the specific forms of G and U.

V. SU(N)-INVARIANT FERMI LIQUID

A. SU(N) vs SU(2)

In this section, we discuss the interplay between the FL- and perturbation theories for a system of interacting fermions with a large number of flavors N. Fermi-liquid properties of such a system were discussed both in terms of RG (Ref. 16) and pertubation theory for the case of a Coulomb interaction.[17,18] In the limit when the N times a (dimensionless) coupling constant is larger than one, renormalization of not only the Z factor but also of the effective mass comes from energies much higher than the Fermi energy. Based on the observation, the authors of Ref. 17 argued that an SU(N)-invariant FL is not of the same type as discussed in the framework of the Landau theory. We show here that this is not the case: the FL- and perturbation theories give the same results for the SU(N)-invariant case as well. To see this, however, one needs to collect next-to-leading terms in the large-N expansion of the FL theory, whereas the perturbation theory can be evaluated only to the leading order in $1/N$.

The difference between the large-N expansions for the FL and perturbation theories is most dramatic for the case of a momentum-independent interaction in 2D and for brevity we consider only this case here. In the SU(N) case, each particle-hole bubble is multiplied by a factor of N. The second-order self-energy contains two diagrams—(c) and (d) in Fig. 2—the first of which acquires a factor of N while the second does not. Consequently, the perturbative effective mass acquires a factor of $N-1$ compared to the result in Eq. (2.13)

$$\frac{m^*}{m} = 1 + \frac{N-1}{2}\left(\frac{mU}{2\pi}\right)^2 \approx 1 + \frac{N}{2}\left(\frac{mU}{2\pi}\right)^2, \quad (5.1)$$

where the last result applies to the large-N limit. On the other hand, the only diagram for Γ^ω which acquires a factor of N—diagram (f) in Fig. 1—does not depend on the angle between the initial fermionic momenta $\mathbf{p}_F$ and $\mathbf{q}_F$ because the static particle-hole bubble is independent of the momentum in 2D for $|\mathbf{p}_F - \mathbf{q}_F| \leq 2p_F$. Therefore, the leading term in the $1/N$ expansion for Γ^ω does not contribute to mass renormalization. To resolve this contradiction, one needs to recall that when deriving the FL result for m^* [Eq. (1.2b)] we divided the trace of Γ^ω by the number of spin components. This is the origin of the factor of 2 in the prefactor A_D. The formula for the SU(N) case is obtained from Eq. (1.2b) simply by replacing 2 in A_D by N; in 2D, we have

$$\frac{1}{m^*} = \frac{1}{m} - \frac{Z^2}{N(2\pi)^2}\sum_{\alpha\beta}\int \Gamma^\omega_{\alpha\beta,\alpha\beta}(p_F,q_F)\frac{\mathbf{p}_F\cdot\mathbf{q}_F}{p_F^2}d\Omega_q. \quad (5.2)$$

Diagrams for direct processes ($\mathbf{p}\rightarrow\mathbf{p}$, $\mathbf{q}\rightarrow\mathbf{q}$) enter Γ^ω with a factor of $\delta_{\alpha\gamma}\delta_{\beta,\delta}$, which becomes equal to one for $\gamma=\alpha$ and $\delta=\beta$. Therefore, the trace of the direct contribution to Γ^ω gives a factor of N^2, which, upon dividing by an overall factor of N in Eq. (5.2), gives an $\mathcal{O}(N)$ contribution to m^*. To obtain an $\mathcal{O}(N)$ term in m^* from the FL theory, one thus needs to collect *all direct* $\mathcal{O}(1)$ diagrams for Γ^ω. On the other hand, diagrams for exchange processes ($\mathbf{p}\rightarrow\mathbf{q}$, $\mathbf{q}\rightarrow\mathbf{p}$) enter Γ^ω with a factor of $\delta_{\alpha\delta}\delta_{\beta\gamma}$, which becomes equal to $\delta_{\alpha\beta}$ for $\gamma=\alpha$ and $\delta=\beta$. The trace of the exchange contribution is of order N, which translates into a subleading, $\mathcal{O}(1)$ term in m^*. Therefore, one can neglect exchange processes in Γ^ω in the large-N limit. A physical reason for this simplification is obvious: since the large-N limit is inherently semiclassical, the Pauli principle becomes irrelevant.

The same conclusion also follows from an identity which involves the generators $\hat{T}$ of the SU(N) group[16]

$$\delta_{\alpha\delta}\delta_{\beta\gamma} = \frac{1}{2}\sum_{a=1}^{N^2-1}\hat{T}^a_{\alpha\gamma}\hat{T}^a_{\beta\delta} + \frac{1}{N}\delta_{\alpha\gamma}\delta_{\beta\delta}. \quad (5.3)$$

The delta symbols on the left (right) occur in exchange (direct) contributions to Γ^ω, correspondingly. Since $\hat{T}^a$ are traceless, it follows immediately that the trace of the exchange contribution to Γ^ω is by a factor of N smaller than the trace of the direct contribution.

Coming back to the second-order Γ^ω for arbitrary N, we need to consider all subleading, $\mathcal{O}(1)$ diagrams in Fig. 1. Diagrams (e) and (g) contain particle-hole bubbles evaluated at $|\mathbf{p}_F - \mathbf{q}_F|$ and, therefore, do not contribute to mass renormalization. Direct particle-particle diagram c contributes an $\mathcal{O}(N)$ term to m^* while its exchange counterpart d contributes an $\mathcal{O}(1)$. A combined contribution of diagrams (c) and (d) is $(N-1)$ times the SU(2) result, which is the same as in Eq. (5.1).

In the SU(2) case, the perturbative regime implies that the interaction is weak in a sense that $mU \ll 1$. Then mU is the only parameter in the problem, and Eq. (5.1) is the leading term in the expansion in this parameter for $N=2$. In the SU(N) case with a large number of flavors, there is a nonperturbation regime, defined by the conditions $1/N \ll Um \ll 1$, where the perturbation theory should and can be resummed to infinite order, and Eq. (5.1) is replaced by Eq. (5.10) below. It is instructive to compare the perturbation theory with the FL formalism in this case. The subsequent analysis will be performed in the Matsubara technique at $T=0$.
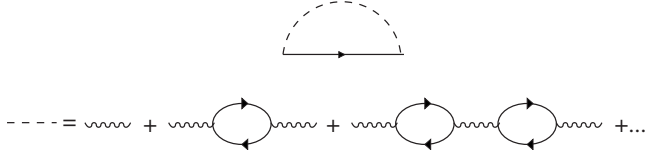
FIG. 3. Self-energy in the large-N limit. The dashed line is an effective interaction given by Eq. (5.5).

B. Pertubation theory for the self-energy in the non-perturbative regime

In the large-N limit, the perturbative self-energy is given diagrams with a maximal number of the particle-hole bubbles at each order in U. The sum of such diagrams is equivalent to a first-order diagram, shown in Fig. 3,

$$\Sigma_{\mathrm{pert}}(\omega, \epsilon_{\mathbf{p}}) = \int \frac{d^2 k}{(2\pi)^2} \int \frac{d\omega_k}{2\pi} G(\omega + \omega_k, \mathbf{p} + \mathbf{k}) \widetilde{U}(\omega_k, \mathbf{k}),$$

(5.4)

where the effective interaction is of an RPA form

$$\widetilde{U}(\omega_k, \mathbf{k}) = \frac{U}{1 - NU\Pi_{\mathrm{ph}}(\omega_k, \mathbf{k})}.$$

(5.5)

As in the SU(2) case, the self-energy can be separated into anomalous and regular part. The anomalous part is similar to Eq. (3.11), except for the second-order effective interaction $U^2\Pi_{\mathrm{ph}}(\omega_k=0, \mathbf{k})$ is replaced by $\widetilde{U}(\omega_k=0, \mathbf{k})$. Still, as $\Pi_{\mathrm{ph}}(\omega_k=0, \mathbf{k})$ is independent of $|\mathbf{k}|$ for $|\mathbf{k}| \leq 2p_F$, the integral over $|\mathbf{k}|$ vanishes and the anomalous part of the self-energy does not renormalize the effective mass.

In the regular part of the self-energy, we assume—and then justify—that the limit of $NUm \gg 1$ corresponds to large momentum transfers: $|\mathbf{k}| \gg p_F$. In this limit, the particle-hole bubble becomes[17]

$$\Pi_{\mathrm{ph}}(\omega_k, \mathbf{k}) = -\frac{p_F^2}{2\pi} \frac{E_{\mathbf{k}}}{\omega_q^2 + E_{\mathbf{k}}^2},$$

(5.6)

where $E_{\mathbf{k}} = \mathbf{k}^2/2m$. At fixed number density of particles n, the area of the Fermi surface is inversely proportional to the number of flavors

$$p_F^2/4\pi = n/N.$$

(5.7)

With this normalization, the product $NU\Pi_{\mathrm{ph}}(\omega_k, \mathbf{k}) = -2nE_{\mathbf{k}}/(\omega_k^2 + E_{\mathbf{k}}^2)$ is independent of N. A pole of $\widetilde{U}$ in real frequencies corresponds to the collective (zero-sound) mode with dispersion $\omega_k = \sqrt{E_{\mathbf{k}}^2 + E_0 E_{\mathbf{k}}}$, which interpolates between soundlike excitations for $E_{\mathbf{k}} \ll E_0 \equiv 2nU$ and particlelike excitations for $E_{\mathbf{k}} \gg E_0$. Note that $E_0/E_F \sim NUm \gg 1$.

According to Eq. (1.3), mass renormalization is determined by a stand-alone $\epsilon_{\mathbf{p}}$ term in the self-energy. This term is obtained by expanding the single-particle dispersion in the argument of the Green's function in Eq. (5.4) as $\epsilon_{\mathbf{p}+\mathbf{k}} = \epsilon_{\mathbf{p}} + \epsilon_{\mathbf{p}} \mathbf{p}_F \cdot \mathbf{k}/p_F^2 + \mathbf{p}_F \cdot \mathbf{k}/m + E_{\mathbf{k}}$ and differentiating the regular part of the self-energy with respect to the second term in $\epsilon_{\mathbf{p}+\mathbf{k}}$ at $\epsilon_{\mathbf{p}} = \omega = 0$. This yields

$$\frac{m}{m^*} = 1 - \int \frac{d^2 k}{(2\pi)^2} \int \frac{d\omega_k}{2\pi} G(\mathbf{p}_F + \mathbf{k}, \omega_k) \widetilde{U}(\omega_k, \mathbf{k}) \frac{\mathbf{p}_F \cdot \mathbf{k}}{p_F^2}.$$

(5.8)

Since, according to our assumption, large $|\mathbf{k}|$ control the integral in Eq. (5.8), the $E_{\mathbf{k}}$ term in the denominator of Green's function is larger than the $\mathbf{p}_F \cdot \mathbf{k}$ term. Expanding the Green's function to first order in $\mathbf{p}_F \cdot \mathbf{k}$, integrating over the angle, and switching from integration over $|\mathbf{k}|$ to integration over $E_{\mathbf{k}}$, we obtain

$$\frac{m}{m^*} = 1 - \frac{Um}{2\pi^2} \int_0^\infty dE_{\mathbf{k}} \int_{-\infty}^\infty d\omega_k \frac{\omega_k^2 + E_{\mathbf{k}}^2}{\omega_k^2 + E_{\mathbf{k}}^2 + E_0 E_{\mathbf{k}}} \frac{E_{\mathbf{k}}}{(i\omega_k - E_{\mathbf{k}})^3}.$$

(5.9)

Now it is obvious that typical $|\omega_k| \sim E_{\mathbf{k}} \sim E_0 \gg E_F$, which justifies our original assumption. Performing remaining integrations, we finally obtain

$$\frac{m^*}{m} = 1 + \frac{mU}{8\pi}.$$

(5.10)

As it is also the case for the Coulomb interaction,[17,18] the effective mass in the $NUm \gg 1$ limit does not depend on N. Note that Eq. (5.10) is valid only for a repulsive and sufficiently weak interaction ($Um \ll 1$) so that mass renormalization is still a small albeit nonperturbative effect. Notice also that the normalization condition (5.7) was not essential: if it is not imposed, the high-energy scale E_0 is replaced by $\widetilde{E}_0 = (NUm/\pi)E_F \gg E_F$. However, since the high-energy scale drops out from the formula (5.9) for the effective mass, changing E_0 by $\widetilde{E}_0$ does not affect the result for m^*.

Differentiating the regular part of the self-energy with respect to $i\omega$ and performing integrations in a way similar to the effective-mass case, we obtain the Z factor in the $NUm \gg 1$ limit

$$Z = 1 - \frac{mU}{4\pi}.$$

(5.11)

C. Fermi-liquid formalism in the non-pertubative regime

As we explained in Sec. V A, to obtain the effective mass in the FL formalism, one needs to collect all *direct* diagrams for Γ^ω to next-to-leading order in $1/N$. This arduous task is simplified dramatically in the case of a momentum-independent interaction in 2D, where diagrams with particle-hole bubbles at $\mathbf{p} - \mathbf{q}$ do not contribute to mass renormalization. To second order in U, the leading order in N is $N^1 = N$ and next-to-leading order is $N^0 = 1$. There is only one direct, $\mathcal{O}(N^0)$ diagram that does not contain $\Pi_{\mathrm{ph}}(\omega=0, \mathbf{p} - \mathbf{q})$—diagram (c) in Fig. 1. To third order in U, there are only two inequivalent diagrams of order $\mathcal{O}(N^1)$, shown in Fig. 4. One of them renormalizes the second-order particle-particle diagram (c) while the other renormalizes the particle-hole diagram (e) in Fig. 1. Note that the particle-hole bubbles in both diagrams are integrated over internal four momenta and, hence, do contribute the angular dependence of Γ^ω. To fourth order in U, there are four inequivalent diagrams, also
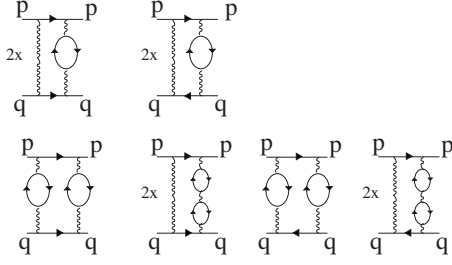
FIG. 4. Diagrams for the Fermi-liquid vertex $\Gamma^\omega_{\alpha\beta,\gamma\delta}(p,q)$ in the large-N limit and to fourth order in the momentum-independent interaction $U=$ const. We select only those diagrams which contribute to mass renormalization in 2D. Note that particle-particle and particle-hole diagrams differ by the directions of arrows on bottom fermionic lines.

shown in Fig. 4. It is easy to see that the overall combinatorial coefficients for particle-particle and particle-hole diagrams of order U^n are both equal to $n-1$. Collecting all orders, we obtain for the angular-dependent part of Γ^ω

$$\Gamma^\omega_{\alpha\beta,\gamma\delta}(p_F,q_F) = -U^2 \delta_{\alpha\beta}\delta_{\gamma\delta}\int \frac{d^3l}{(2\pi)^3}[G_{p+l}G_{q+l}$$
$$+ G_{p+l}G_{q-l}]\sum_{n=0}^{\infty}(n+1)[NU\Pi_{\mathrm{ph}}(l)]^n$$

$$= -U^2 \delta_{\alpha\beta}\delta_{\gamma\delta}\int \frac{d^3l}{(2\pi)^3}[G_{k+l}G_{q+l}$$
$$+ G_{k+l}G_{q-l}]\frac{1}{[1-NU\Pi_{\mathrm{ph}}(l)]^2}, \qquad (5.12)$$

where we added the $\mathcal{O}(U^2)$ particle-hole diagram, which does not contribute to mass renormalization, to the right-hand side of Eq. (5.12). The result for $\Gamma^\omega_{\alpha\beta,\gamma\delta}$ is greatly simplified in the limit of $NUm \gg 1$. As in the previous section, we replace $\Pi_{\mathrm{ph}}(l)$ by its large-momentum asymptotic form Eq. (5.6) and expand the products of Green's functions as

$$G_{p+l}G_{q+l} = \frac{(v_F l)^2}{(i\omega_l - E_\mathbf{l})^4}\cos\theta_{pl}\cos\theta_{ql} + \cdots,$$

$$G_{p+l}G_{q-l} = -\frac{(v_F l)^2}{(\omega_l^2 + E_\mathbf{l}^2)^2}\cos\theta_{pl}\cos\theta_{ql} + \cdots, \quad (5.13)$$

where $\cdots$ stand for angular-independent and higher order terms, $\theta_{k_1 k_2} \equiv \angle(\mathbf{k}_1,\mathbf{k}_2)$, and, as before, $E_\mathbf{l} = l^2/2m$. After trivial angular integration and some simplifications, we obtain

$$\Gamma^\omega_{\alpha\beta,\gamma\delta}(p_F,q_F) = -\delta_{\alpha\beta}\delta_{\gamma\delta}\frac{U^2 k_F^2}{\pi^2}\cos\theta_{pq}\int_0^\infty$$

$$\times dE_\mathbf{l}\int_{-\infty}^\infty d\omega_l \frac{i\omega_l E_\mathbf{l}^2}{(i\omega_l - E_\mathbf{l})^2}\frac{1}{(\omega_l^2 + E_\mathbf{l}^2 + E_0 E_\mathbf{l})^2}$$

$$= \delta_{\alpha\beta}\delta_{\gamma\delta}\cos\theta_{pq}\frac{U}{2N}. \qquad (5.14)$$

Substituting Eq. (5.14) into the formula for the effective mass [Eq. (5.1)], we reproduce the result of the perturbation theory, Eq. (5.10).

To reproduce the perturbative result for Z, one needs to evaluate $\Gamma^\omega_{\alpha\beta,\gamma\delta}(p_F,q)$ for q away from the FS. We did not attempt to do this.

VI. CONCLUSIONS

In this paper, we analyzed an interplay between high- and low-energy contributions to two fundamental Fermi-liquid parameters—the quasiparticle Z factor and effective mass m^*—obtained in two different ways: via a general FL formalism and via a diagrammatic perturbation theory. In both cases, Z and m^* are extracted from the fermionic self-energy. In the FL formalism, the self-energy Σ_{FL} is obtained from the Pitaevskii identities for the derivatives of the Green's functions (following from the particle number conservation and Galilean invariance) and expressed via an antisymmetrized FL vertex Γ^ω. In the perturbation theory, the self-energy Σ_{pert} is obtained in series of nonantisymmerized interaction $U(|\mathbf{k}|)$. To any order in $U(|\mathbf{k}|)$, the two self-energies are not identical when expressed in terms of fermionic Green's functions but certainly yield the same expressions for m^*/m and Z. We found, however, that identical results for m^* in the two approaches are determined by different regions of energies. Whereas the FL-theory m^* comes from low energies, i.e., from the vicinity of the Fermi surface, the perturbative m^* includes, in general, contributions from both low and high energies. Only the sum of the two contributions coincides with the FL result for m^*. We found that the equivalence of m^*/m in the two approaches is based on a particular identity for the products of fermionic Green's functions, Eq. (4.6) which relates the low- and high-energy contributions to the effective mass. On the other hand, renormalization of Z comes only from high-energy fermions in both approaches. We obtained the expression for Z in a 2D Fermi liquid with short-range interaction.

We also analyzed the difference between the FL- and perturbative approaches for a system of interacting fermions with SU(N) symmetry in the limit of $N \gg 1$. We showed that mass renormalization in the diagrammatic formalism comes from high energies and that equivalent expressions for the effective mass are obtained in the two formalisms only if one collects next-to-leading terms in the $1/N$ expansion for Γ^ω. We obtained a closed expression for Γ^ω for the case of momentum-independent interaction in 2D.

The lesson to be learned from this consideration is that

one has to be careful with eliminating high-energy fermions from the problem. While it is tempting to reduce the problem to that of low-energy fermions with an effective interaction and consider only the low-energy contribution to m^*/m, this would give an incorrect result for m^*. The reason is that, in the process of integrating out high-energy fermions, the quasiparticle mass changes from its bare value, m, to a new one, m_B. The difference $m_B/m-1$ comes from high energies. Only the combined effect of high-energy renormalization, which replaces m by m_B, and low-energy renormalization, which involves only the Fermi-surface states, yields the agreement with the FL theory.

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[14] See, e.g., V. M. Galitski and S. Das Sarma, Phys. Rev. B **70**, 035111 (2004), and references therein.

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